

Lévy flights in inhomogeneous environments

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We study the long time asymptotics of probability density functions (pdfs) of Lévy flights in different confining potentials. For that we use two models: Langevin - driven and (Lévy - Schrödinger) semigroup - driven dynamics. It turns out that the semigroup modeling provides much stronger confining properties than the standard Langevin one. Since contractive semigroups set a link between Lévy flights and fractional (pseudo-differential) Hamiltonian systems, we can use the latter to control the long - time asymptotics of the pertinent pdfs. To do so, we need to impose suitable restrictions upon the Hamiltonian and its potential. That provides verifiable criteria for an invariant pdf to be actually an asymptotic pdf of the semigroup-driven jump-type process. For computational and visualization purposes our observations are exemplified for the Cauchy driver and its response to external polynomial potentials (referring to Lévy oscillators), with respect to both dynamical mechanisms.

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I. INTRODUCTION

In recent years there has been growing interest in random walks, extending from various fields of physics to chemistry, biology and financial mathematics. The classical concept of Brownian motion has become paradigmatic in the whole theory of stochastic processes, see e.g. Ref. [1]. The probability density function (pdf) of a homogeneous Brownian motion solves a Fokker-Planck equation and has an important intrinsic property: the diffusing particle pdf, that is initially concentrated at a point, with the flow of time takes the Gaussian form, whose width grows in time as $t^{1/2}$. This kind of diffusion processes was called the normal diffusion. We leave aside a broad field of anomalous diffusions, where $\langle X^2(t) \rangle \sim t^\alpha$ with $0 < \alpha < 2$, and focus on non-Gaussian, jump-type stochastic processes, whose pdfs belong to a class of Lévy - stable distributions. They have long tails and their second (and higher) moments are nonexistent. The existence of the first moment is granted only for a suitable subclass [2].

In the standard theory of diffusion-type processes, a celebrated method to solve transport problems with respect to the pdf and specifically transition probability density, is to transform the involved Fokker-Planck equation into its parabolic (Hermitian) counterpart, [3]. Right at this point a dynamical (Schrödinger) semigroup notion naturally appears and its obvious link with a Hamiltonian dynamical system. Indeed, one encounters here the Schrödinger-type equation (there is no imaginary unit before time derivative and physical dimensions of the Hamiltonian are re-scaled accordingly), often called the generalized diffusion equation, $\partial_t \Psi = \hat{H} \Psi$. Here, \hat{H} is interpreted as a Hamiltonian operator and $-\hat{H}$ stands for a generator of the dynamical semigroup $\exp(-t\hat{H})$.

In case of Brownian motion and normal diffusion process, the Fokker - Planck (Langevin - driven) and semigroup dynamics refer to the same random process. A usefulness of the the semigroup picture lies in the possibility of eigenfunction expansions of \hat{H} which allows to deduce

explicit formulas for transition (semigroup) kernels and transition pdfs, [3]. As a byproduct of the above procedure one solves the eigenvalue problem for the Hamiltonian operator and identifies its ground state as a square root of an invariant pdf of the stochastic diffusion process. The latter pdf is approached by the process at large times.

The situation is somewhat different in the case of non-Gaussian jump-type processes. Namely, the (Lévy - Schrödinger) semigroup dynamics differs from the Langevin - driven fractional Fokker-Planck evolution, see e.g. [4, 5, 6]. Nonetheless a common asymptotic invariant pdf may be attributed to both dynamical scenarios [4]. Since Langevin-driven dynamics does not admit invariant pdfs in the Gibbs form (e.g. the force potential does not appear in the exponential form of ρ_*), the whole class of pdfs, originally employed in the study of topologically induced super-diffusions [5, 6], might seem to be excluded by the formalism of Ref. [4]. This is not the case.

In the present paper we focus on extending the range of validity of the reverse engineering (targeted stochasticity) problem of Ref. [7] to so-called topological Lévy processes (topologically induced super-diffusions), occurring in systems with topological complexity like folded polymers and complex networks. To be more specific, the original reverse engineering problem formulates as follows: given an invariant pdf $\rho_*(x)$, design a stochastic Langevin-driven jump-type process for which the preselected density may be an *asymptotic* target. The basic reconstruction goal is to deduce the drift function of the process.

In the previous paper, [4], we have recast the reverse engineering problem so that the original task was supplemented by one more reconstruction step. Namely, we have addressed the existence issue of the Lévy - Schrödinger semigroup potential $\mathcal{V}(x) = -\lambda(|\Delta|^{\mu/2} \rho_*^{1/2})(x) / \rho_*^{1/2}(x)$ (see below for detailed explanation), given the *very same* (as for the Langevin process) invariant pdf $\rho_*(x)$, that is non-Gibbsian by con-

struction.

Presently, we relax the previous (common pdf) constraint and address a fully fledged reconstruction problem for the semigroup dynamics: given an invariant pdf, identify the semigroup-driven Lévy process for which the prescribed invariant pdf ρ_* may stand for an asymptotic one. Then, Gibbsian densities appear to be admissible and a class of jump-type processes, that respond to environmental inhomogeneities, becomes largely extended. The corresponding jump-type processes are identified by us as topological processes due to their links with topologically-induced super-diffusions, [5, 6].

At this point let us stress that it has never been settled that the invariant pdfs of a topological process actually are the *proper asymptotic* ones, e.g. can be reached in the large time asymptotics irrespective from a particular choice of initial data. To this end one must resort to the contractive semigroup notion. One of the aims of our present discussion is to carefully check this point.

The expected asymptotic behavior may not persist for an unrestricted initial pdf choice. The signature, of whether such behavior is allowed or prohibited by the semigroup dynamics, is encoded in the functional form of a semigroup potential $\mathcal{V}(x)$. The latter needs to be reconstructed from the target pdf $\rho_*(x)$, by means of the above generalized reverse engineering problem and respect a number of restrictions. Minimal requirements upon the associated pseudo-differential Hamiltonian and its potential $\mathcal{V}(x)$ were set in Ref. [10], where an explicit construction of Cauchy semigroups has been carried out. First explicit examples of appropriate potentials were found in [4]. It is the contractive semigroup dynamics that guarantees a proper asymptotic behavior of inferred time-dependent pdfs, c.f. also [8] for a more advanced exposition of that issue.

II. LÉVY SEMIGROUPS IN A RANDOM MOTION

A. Brownian pre-requisites

If we have a one-dimensional Smoluchowski diffusion process [3] with an initial pdf $\rho_0(x)$, then its time evolution is determined by the Fokker-Planck equation $\partial_t \rho = D\Delta\rho - \nabla(b \cdot \rho)$ where D is a diffusion coefficient and the time - independent drift $b(x) = f(x)/m\beta = -(1/m\beta)\nabla V(x)$ is induced by an external (conservative, Newtonian) force field $f(x) = -\nabla V(x)$. We adopt a standard form $D = k_B T/m\beta$ of the diffusion coefficient, where m and β are, respectively, a mass and a reciprocal relaxation time of a particle.

Following a standard procedure [3] we may identically rewrite the Fokker-Planck equation in terms of an associated Hermitian (Schrödinger-type) problem by means of a redefinition

$$\rho(x, t) = \Psi(x, t) \rho_*^{1/2}(x) \quad (1)$$

that takes the Fokker-Planck equation into a parabolic one, often called a generalized diffusion equation:

$$\partial_t \Psi = D\Delta\Psi - \mathcal{V}\Psi \quad (2)$$

for a positive function $\Psi(x, t)$. The auxiliary potential \mathcal{V} derives from a compatibility condition $\mathcal{V}(x) = D\Delta\rho_*^{1/2}/\rho_*^{1/2}$, whose equivalent form reads $\mathcal{V}(x) = (1/2)[b^2/(2D) + \nabla b]$.

If the $(1/2mD)$ rescaled) Schrödinger-type Hamiltonian $\hat{H} = -D\Delta + \mathcal{V}$ is a self-adjoint operator in a suitable Hilbert space, then one arrives at a dynamical semigroup $\exp(-t\hat{H})$. We note here, that the Schrödinger semigroup (parabolic) reformulation of the Fokker - Planck equation is merely another mathematical "face" of the diffusion process, the operator \hat{H} is just one more form of the Fokker-Planck operator [5]. The semigroup is contractive, hence asymptotically $\Psi(x, t)|_{t \rightarrow \infty} \rightarrow \rho_*^{1/2}(x)$. Accordingly, $\rho(x, t)|_{t \rightarrow \infty} \rightarrow \rho_*(x)$.

We note that for $\mathcal{V} = \mathcal{V}(x)$ bounded from below, the integral kernel $k(y, s, x, t) = \{\exp[-(t-s)\hat{H}]\}(y, x)$, $s < t$, of the dynamical semigroup $\exp(-t\hat{H})$, is positive [9]. The semigroup dynamics reads: $\Psi(x, t) = \int \Psi(y, s) k(y, s, x, t) dy$ so that for all $0 \leq s < t$

$$\rho(x, t) = \rho_*^{1/2}(x) \Psi(x, t) = \int p(y, s, x, t) \rho(y, s) dy, \quad (3)$$

where

$$p(y, s, x, t) = k(y, s, x, t) \frac{\rho_*^{1/2}(x)}{\rho_*^{1/2}(y)} \quad (4)$$

is the transition probability density of the pertinent Markov process. Its unique asymptotic invariant pdf is $\rho_*(x)$.

For the familiar Ornstein-Uhlenbeck version of the Smoluchowski process, the drift is a linear function of x , e.g. $b(x) = -\gamma x$, $\gamma \equiv \kappa/m\beta$, $\kappa > 0$. The Fokker - Planck equation $\partial_t \rho = D\Delta\rho + \gamma \nabla(x\rho)$ supports an invariant density

$$\rho_*(x) = \left(\frac{\gamma}{2\pi D}\right)^{1/2} \exp\left(-\frac{\gamma}{2D}x^2\right) = \exp\left(\frac{F_* - V(x)}{k_B T}\right),$$

where $V(x) = \kappa \frac{x^2}{2}$ and $F_* = -k_B T \ln(2\pi k_B T/\kappa)^{1/2}$. The associated generalized heat equation involves $\hat{H} = -D\Delta + \mathcal{V}$ with $\mathcal{V}(x) = \frac{\gamma^2 x^2}{4D} - \frac{\gamma}{2}$ which is a typical confining potential. Accordingly, $\hat{H}\rho_*^{1/2} = 0$. The lowest eigenvalue 0 of this positively defined operator identifies $\rho_*^{1/2}$ as its ground state function. Eq. (4) rewrites as

$$p(y, s, x, t) = k(y, s, x, t) \exp[V(y) - V(x)]/2k_B T. \quad (5)$$

B. Free Lévy-Schrödinger Hamiltonian

To consider the properties of a free (without external potentials) Lévy-Schrödinger semigroup, we employ the

rescaled Hamiltonians rather than semigroup generators that have an opposite sign. The pertinent Hamiltonians have the form $\hat{H} = F(\hat{p})$, where $\hat{p} = -i\nabla$ stands for the momentum operator (up to the scaled away \hbar or $2mD$ factor), and for $-\infty < k < +\infty$, the function $F = F(k)$ is real valued and bounded from below. The action of $\exp(-t\hat{H})$ can be given by means of an integral kernel $k_t \equiv k(x-y, t) = k(y, 0, x, t)$ where $k_t(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[-tF(p) + ipz] dp$.

Our further discussion is limited to non - Gaussian random variables whose pdfs are centered and symmetric, e.g. to a subclass of stable distributions characterized by

$$F(p) = \lambda|p|^\mu \Rightarrow \hat{H}_\mu \equiv \lambda|\Delta|^\mu. \quad (6)$$

Here $0 < \mu < 2$ and $\lambda > 0$ stands for an intensity parameter of Lévy process. To account for the interval $0 \leq \mu \leq 2$ boundaries, one should rather employ $(-\Delta)^\mu$ instead of $|\Delta|^\mu$; $-\Delta$ is a positive operator.

The pseudo-differential Hamiltonian \hat{H}_μ , by construction is positive and self-adjoint on a properly tailored domain. A sufficient and necessary condition for both these properties to hold true is that the pdf of the Lévy process is symmetric, see Ref. [13]. The corresponding contractive semigroup admits an analytic continuation in time leading to Lévy-Schrödinger equations and fractional quantum mechanics [14, 15, 16]. The associated jump-type dynamics is interpreted in terms of Lévy flights. The pseudo-differential Fokker-Planck equation, which corresponds to the free fractional Hamiltonian \hat{H}_μ and the fractional semigroup $\exp(-t\hat{H}_\mu) = \exp(-\lambda|\Delta|^\mu)$, reads

$$\partial_t \rho = -\lambda|\Delta|^\mu \rho, \quad (7)$$

to be compared with the ordinary heat equation $\partial_t \rho = +D\Delta\rho$. In particular $F(p) = \lambda|p|$ refers to Cauchy process.

The action of the pseudo-differential operator $|\Delta|^\mu$ on a function can be expressed by the formula [10, 12]

$$(|\Delta|^\mu f)(x) = - \int [f(x+y) - f(x)] \nu_\mu(dy), \quad (8)$$

where $\nu_\mu(dy)$ is a corresponding Lévy measure (see, e.g. [2]) and the integral in Eq. (8) is understood in a sense of its Cauchy principal value. Changing the integration variable y to $z = x+y$ and employing a definition of Riesz fractional derivative of the μ -th order, [18], we arrive at

$$(|\Delta|^\mu f)(x) = - \frac{\Gamma(\mu+1) \sin(\pi\mu/2)}{\pi} \int \frac{f(z) - f(x)}{|z-x|^{1+\mu}} dz \quad (9)$$

with $(|\Delta|^\mu f)(x) = -\partial^\mu f(x)/\partial|x|^\mu$. The case of $\mu = 1$ refers to the Cauchy driver (e.g. noise). We note a systematic sign difference between our notation for a pseudo-differential operator $|\Delta|^\mu$ and that based on the fractional derivative notion, like e.g. $\Delta^{\mu/2} \doteq \partial^\mu/\partial|x|^\mu$ of Refs. [5, 6].

C. Response to external potentials

Consider now the Lévy-Schrödinger Hamiltonian with external potential

$$\hat{H}_\mu \equiv \lambda|\Delta|^\mu + \mathcal{V}(x). \quad (10)$$

Suitable properties of \mathcal{V} need to be assumed, so that $-\hat{H}_\mu$ is a legitimate generator of a dynamical semigroup $\exp(-t\hat{H}_\mu)$, see e.g. Ref. [10].

Looking for stationary solutions of the equation $\partial_t \Psi = \hat{H}_\mu \Psi$, we realize that if a square root of a positive invariant pdf $\Psi \sim \rho_*^{1/2}$ is asymptotically to come out, then fractional Sturm-Liouville operator should be used to derive an explicit form of $\rho_*^{1/2}$ for a given \mathcal{V} . In the opposite situation, when $\rho_*(x)$ is a priori prescribed, we can determine \mathcal{V} through a compatibility condition:

$$\mathcal{V} = -\lambda \frac{|\Delta|^\mu \rho_*^{1/2}}{\rho_*^{1/2}}. \quad (11)$$

The main point here is that we do not have too much freedom in pre-selecting a functional form of ρ_* , as suitable conditions need to be respected by the inferred auxiliary potential \mathcal{V} , to yield a contractive semigroup dynamics. This leads to a conclusion [10] that only under the contractive semigroup premises, an invariant pdf of a jump-type process may actually become its asymptotic target. The detailed discussion of this issue for Cauchy semigroups can be found in Ref. [10].

To derive the pseudo-differential equation governing the behavior of a system in an external potential, we rewrite the pdf of the semigroup-driven stochastic process in the form (1) (see also [4]). Any strictly positive function (here we consider only functions that vanish for large x) can be rewritten in an exponential form. Hence, by adopting the notation $\rho_*(x) = \exp[2\Phi(x)]$ and accounting for (11) we arrive at a continuity equation with an explicit fractional input

$$\partial_t \rho = -\lambda(\exp \Phi)|\Delta|^\mu [\exp(-\Phi)\rho] + \mathcal{V} \rho. \quad (12)$$

The definition (11) suggests that *any* pdf of the form $\rho_*(x) = \exp[2\Phi(x)]$ is a proper candidate for a stationary solution of the Eq. (12). However things are not that simple. It is only the semigroup dynamics generated by (11) and (12) that may guarantee a consistent temporal approach towards an asymptotic invariant density of the stochastic process in question. Right at this point, an issue of restrictions upon an effective potential \mathcal{V} of Eq. (12), [4, 10], enters the stage.

It is instructive to mention that for Lévy flights in external force fields, the (somewhat left aside) Langevin approach is known to yield [17] a continuity (e.g. fractional Fokker-Planck) equation in a very different form

$$\partial_t \rho = -\nabla \left(-\frac{\nabla V}{m\beta} \rho \right) - \lambda|\Delta|^\mu \rho. \quad (13)$$

Even if we know [4], that an asymptotic invariant density of Eq. (12) may coincide with that for Eq. (13), these two transport equations refer to different temporal patterns of behavior.

We note that, in contrast to the semigroup modeling, the Langevin scenario for Lévy flights in confining potentials has received ample attention in the literature, see [7, 17, 18, 19, 20] to cite a few.

D. Topologically induced super-diffusions

It is of some interest to invoke an independent (so-called topological, see above) approach of Refs. [5, 6] where one modifies jumping rates by suitable local factors to arrive at a response mechanism that is characteristic of the previously outlined semigroup dynamics. Namely, in view of (9), the free transport equation (7) can be re-written as a master equation:

$$\partial_t \rho(x) = \int [w(x|z)\rho(z) - w(z|x)\rho(x)]\nu_\mu(dz). \quad (14)$$

The jump rate is an even function, $w(x|z) = w(x|z)$. However, if we replace the jump rate

$$w(x|y) \sim 1/|x - y|^{1+\mu} \quad (15)$$

(c.f. Eq. (9)) by the expression

$$w_\phi(x|y) \sim \frac{\exp[\Phi(x) - \Phi(y)]}{|x - y|^{1+\mu}} \quad (16)$$

and account for the fact that $w_\phi(x|z) \neq w_\phi(z|x)$, the corresponding transport equation takes the form:

$$(1/\lambda)\partial_t \rho = |\Delta|_\Phi^{\mu/2} f = -\exp(\Phi) |\Delta|^{\mu/2} [\exp(-\Phi)\rho] + \rho \exp(-\Phi) |\Delta|^{\mu/2} \exp(\Phi). \quad (17)$$

Whatever potential $\Phi(x)$ has been chosen (up to a normalization factor), then formally $\rho_*(x) = \exp(2\Phi(x))$ is a stationary solution of Eq. (17). Moreover, one readily verifies that Eq. (17) is identical with the semigroup-induced Eq. (12).

Accordingly, if for a pre-determined $\rho_* = \exp(2\Phi)$, there exists the semigroup potential \mathcal{V} , Eq. (11), then Eq. (17) defines the dynamics that belongs to the previously outlined semigroup framework. This entails a direct verification of whether the stationary density ρ_* may really be interpreted as an asymptotic target of the pertinent fractional transport equation.

Stationary pdfs for the topologically-induced dynamics were demanded to occur in a Gibbsian form $\exp(2\Phi)$, see [5, 6] for a possible phenomenological background for this assumption. Therefore, we redefine Φ as follows. Rewriting the stationary pdf ρ_* as $\rho_*(x) = (1/Z) \exp(-V_*(x)/k_B T)$ (normalization factor Z stands for a partition function), we recover a function $V_*(x) = -k_B T \ln(Z \rho_*(x))$ that receives an interpretation of the external potential. With these re-definitions, the previous equation (17) takes the customary form employed in the discussion of topologically induced super-diffusions:

$$\partial_t \rho = -\exp(-\kappa V_*/2) |\Delta|^{\mu/2} \exp(\kappa V_*/2) \rho + \rho \exp(\kappa V_*/2) |\Delta|^{\mu/2} \exp(-\kappa V_*/2), \quad \kappa = 1/k_B T. \quad (18)$$

III. RESPONSE OF CAUCHY NOISE TO POLYNOMIAL POTENTIALS

A. Ornstein - Uhlenbeck - Cauchy process

In case of the Ornstein - Uhlenbeck - Cauchy (OUC) process, the drift is given by $b(x) = -\gamma x$, and an asymptotic invariant pdf associated with

$$\partial_t \rho = -\lambda |\nabla| \rho + \nabla[(\gamma x) \rho] \quad (19)$$

reads:

$$\rho_*(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}, \quad (20)$$

where $\sigma = \lambda/\gamma$, c.f. Eq. (9) in Ref. [11].

Note that a characteristic function of this density reads $F(p) = -\sigma|p|$ and gives account for a non - thermal fluctuation - dissipation balance. The modified noise intensity parameter σ is a ratio of an intensity parameter λ of the Cauchy noise and of the friction coefficient γ .

The invariant density of OUC process (20) generates (with the help of Eq. (11)) the following Cauchy semigroup potential \mathcal{V} (we set $\mu = 1$)

$$\mathcal{V}(x) = \frac{\lambda}{\pi} \left[-\frac{2}{\sqrt{a}} + \frac{x}{a} \ln \frac{\sqrt{a} + x}{\sqrt{a} - x} \right], \quad (21)$$

$$a = \sigma^2 + x^2.$$

The potential (21) had been analyzed in Ref. [4]. It is clear that $\mathcal{V}(x)$ is bounded both from below and above, well fitting to the general mathematical construction of (semigroup-driven or topological) Cauchy processes in external potentials, [10].

Since the OUC pdf (20) has no variance, in Fig. 1 we visualize the temporal evolution of OUC process with initial data localized at $x = 0$ ($\rho(x, t = 0) = \delta(x)$) in two motion scenarios (i.e. Langevin and semigroup driven) by comparing the width of the OUC "bell" at its half-maximum at a number of consecutive instants of time. It is seen that Langevin dynamics sets at equilibrium faster

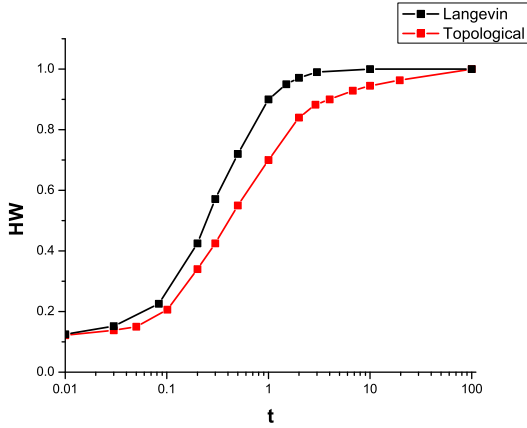


FIG. 1: Temporal behavior of the half-maximum width (HW): for the OUC process in Langevin-driven and semigroup-driven (topological) processes. Motions begin from common initial data $\rho(x, t = 0) = \delta(x)$ and end up at a common pdf (20) for $\sigma = 1$.

than the semigroup-induced dynamics.

Further check of the OUC dynamics is to obtain (numerically [21]) the time evolution $\rho(x, t)$ of initial data, localized in another point. Since quadratic potential (corresponding to OUC process) has a minimum at $x = 0$, we investigate the temporal evolution for the initial data that are shifted from the point $x = 0$. Namely, we consider two cases - "left" $\rho_L(x, t = 0) = \delta(x + 1)$ and "right" $\rho_R(x, t = 0) = \delta(x - 1)$ ones. As the "left" and "right" cases are symmetric with respect to y axis, in Fig. 2 we report the "left" case only. It is seen that final stage of the evolution is still the invariant pdf (20). This means that fictitious particle representing our process, "rolls down" to potential minimum at $x = 0$. We note that this "rolling down" occurs slower than the evolution with initial data, localized at $x = 0$. The topologically-induced dynamics of $\rho_{L,R}(x, t = 0)$ is qualitatively the same, except for the just mentioned slow-down, as Langevin-induced one (see Ref. [4]) and we do not show it here.

A pre-selection of the OUC ρ_* , Eq. (20), proved to provide (through a numerical process reconstruction) a consistent asymptotic target pdf for the Cauchy semigroup-driven dynamics. At this point we may justifiably ask about limitations upon freedom of choice that is admitted in such invariant pdf pre-selection procedure.

Remark 1: As a byproduct of the discussion we have established a pseudo-differential Hamiltonian system, whose ground state equals the square root $\rho_*^{1/2}(x)$ of the Cauchy pdf (20).

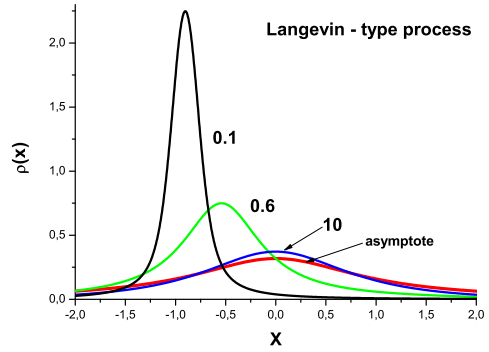


FIG. 2: Time evolution of Langevin-driven pdf $\rho_L(x, t)$ beginning from the initial data $\rho_L(x, t = 0) = \delta(x + 1)$ and ending at the pdf (20) (shown as "asymptote" in the figure) for $\sigma = 1$. Figures near curves correspond to t values.

B. Cauchy driver: Polynomial drifts and the semigroup evolution

The OUC case corresponds to a linear drift function. A number of polynomial drift functions has been discussed in the literature with a focus on confining features of various external forces on Lévy flights. In each case a corresponding asymptotic invariant pdf has been found, albeit with a restriction (in view of limited computational facilities) to the Cauchy driver. For clarity of subsequent discussion below we provide some examples.

The quadratic Cauchy density $\rho_*(x) = 2/\pi(1 + x^2)^2$ stands for an asymptotic target of the Langevin - driven process with the drift $b(x) = (-\gamma x/8)(x^2 + 3)$ [4]. For the above quadratic Cauchy pdf, the associated semigroup evolution is defined, by means of the potential function $\mathcal{V}(x) = \lambda(x^2 - 1)/(x^2 + 1)$, obtained from (11). This potential is also bounded from below and above and consequently, [10], defines a contractive semigroup generator and hence semigroup dynamics.

The bimodal pdf $\rho_*(x) = \beta^3/\pi(x^4 - \beta^2 x^2 + \beta^4)$ can be derived from the drift function $b(x) = -\gamma x^3/\beta^3$. The associated semigroup dynamics is correctly defined by means of a potential function \mathcal{V} , obtained from Eq. (11) numerically. Its shape is reported in Fig. 3. It is seen, that the potential is bounded from below and above, again perfectly fitting to the general theory of [10].

C. Cauchy driver: Gibbsian versus non-Gibbsian asymptotics

The Langevin-driven dynamics with a given drift $b \sim -\nabla V$, where $-\nabla V$ stands for a conservative force acting upon a particle in the course of its random motion, for non-Gaussian driver (like e.g. Cauchy one) does not admit an asymptotic invariant density in the Gibbsian form

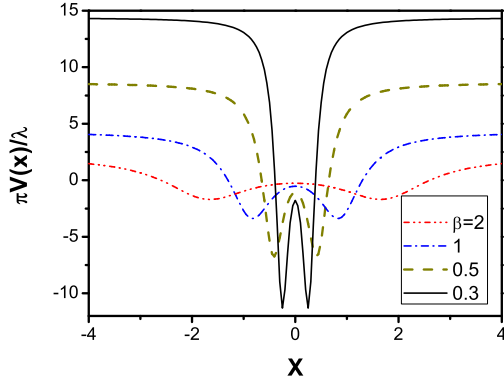


FIG. 3: The coordinate dependence of $\mathcal{V}(x)$ associated with the bimodal pdf, for different values of β .

$$\rho_* \sim \exp(-V/k_B T), [7].$$

As we have established above, see also [4], the Langevin-driven and semigroup-driven jump-type processes, with the Cauchy driver, may share common asymptotic stationary pdfs, that are obviously non-Gibbsian.

On the other hand, as mentioned in Section II.D, one may suspect that asymptotic invariant pdfs of a semigroup-driven process (in other words, topologically-induced one) may not necessarily have fat (e.g. non-Gaussian) tails, due to extremely strong confinement of admissible jumps, imposed by the "potential landscape" of an inhomogeneous medium. That amounts to assuming that a Gibbs-type asymptotic pdf $\rho_* = (1/Z) \exp(-V_*/k_B T)$ may be employed in the construction of the topologically-induced dynamics (18).

A thermalization mechanism, under which the equilibrium conditions could have been achieved for such Gibbsian ρ_* (the non-Gaussian mechanism is excluded [7] and has not been considered in Refs. [5, 6]) is actually unclear. The major focus in the literature was upon a suitable "potential landscape" (potential profile), such that local modifications (16) of the jump rates would drive the random motion towards a Gibbsian equilibrium.

We shall follow the "potential landscape" intuition and refer to explicit dynamics simulations reported in Ref. [5] for the double well potential $V_*(x) \equiv \Phi(x) = x^4 - 2x^2 + 1$. Dimensional units are scaled away and, to facilitate comparison, the notation Φ refers presently to that of Ref. [5]), see Fig. 3 therein. As a supplementary test, we consider also $\Phi \equiv V_*(x) = x^2$, leading to Gibbsian asymptotic pdf in the Gaussian form.

Our next step was to evaluate (numerically) the semigroup potential \mathcal{V} , after literally substituting a concrete Gibbsian asymptotic pdf ρ_* to Eq. (11). The outcome is reported in Fig. 4.

The semigroup potentials \mathcal{V} depicted in Fig. 4 are

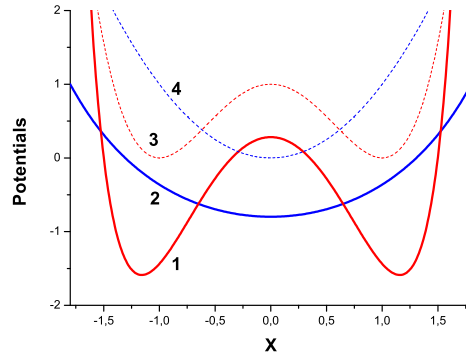


FIG. 4: The coordinate dependence of the semigroup potential $\mathcal{V}(x)$ (curves 1 and 2), corresponding to $V_*(x) = x^4 - 2x^2 + 1$ (curve 3) and $V_*(x) = x^2$ (curve 4), respectively. Curves 3 and 4 are shown for a comparison with, strikingly similar in shape, semigroup potential curves 1 and 2

admitted by the general theory of Ref. [10] (see below for details), so granting the operator $\exp(-t\hat{H})$ ($\hat{H} = -|\nabla| - \mathcal{V}$), the contractive semigroup status. This proves that an invariant (stationary) density of Eq. (19), with the bimodal or harmonic exponent, actually is an *asymptotic invariant* pdf of the topological process.

The topological confining mechanism appears to be much stronger than confining mechanism based on the Langevin modelling. Namely, if both mechanisms share the same potential $\Phi(x) = V_*(x)$, then the asymptotic pdf in Langevin scenario has no more than a finite number of moments. At the same time, the corresponding asymptotic pdf of the topological process, being in the form $\propto \exp(-\Phi)$, under the Gibbsian premises may in principle admit all moments. To the contrary, if we impose for both mechanisms to have the same asymptotic pdf, the above statement is invalid.

For completeness, we now recollect the above mentioned requirements upon \mathcal{V} that need to be observed in the presence of Cauchy driver, [10]. Namely, the potential should allow to be made positive (by merely adding a constant), should be locally bounded and needs to be measurable (i.e. can be approximated with arbitrary precision by step functions sequences). The Cauchy generator plus a potential with such properties is known to determine uniquely [10] an associated Markov process of the jump-type and its step functions approximation. The limiting behavior of the pertinent step process, as the step size is going to zero, remains under control.

Remark 2: Would we have followed the standard Langevin modeling for the Cauchy driver, with the external force potential $V_*(x) = x^4 - 2x^2 + 1$ and the resultant drift $-\nabla V_* = b$, an invariant pdf of the corresponding

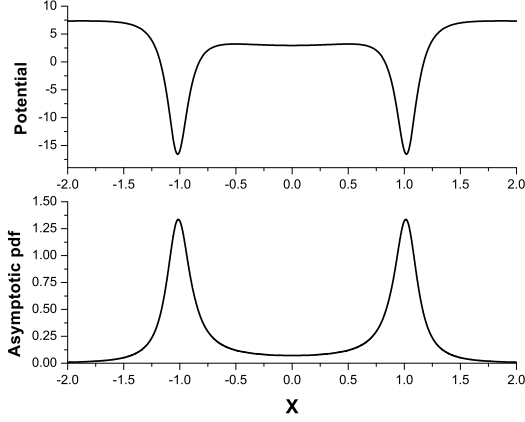


FIG. 5: The coordinate dependence of the semigroup potential $\mathcal{V}(x)$ derived from the non-Gibbsian pdf of Remark 2, Section III.C, along with the pdf (22).

fractional Fokker-Planck equation would have the form:

$$\rho_*(x) = \frac{2a(a^2 + b^2)}{\pi} \frac{1}{(a^2 + b^2)^2 + 2(a^2 - b^2)x^2 + x^4} \quad (22)$$

with $a \simeq 0.118366$ and $b \simeq 1.0208$. Here, a and b are, respectively, real and imaginary parts of complex roots of the cubic equation $z^3 + z - 1/4 = 0$. It is easy to show both analytically and numerically that the above ρ_* is properly normalized. By using the formula (11) we may associate with the non-Gibbsian pdf a semigroup potential. Its properties prove that we deal with a contractive semigroup dynamics and a common asymptotic pdf for both Langevin and semigroup motion scenarios. The outcome of this discussion is depicted in Fig 5.

Remark 3: Standard arguments, c.f. Section II.A, convince us that the Gibbsian form $\rho_* \sim \exp(-V_*/k_B T)$ of an asymptotic density would have been recovered in the presence of the Wiener (Brownian) noise, given the drift function $b \sim -\nabla V_*$ with $V_*(x) = x^4 - 2x^2 + 1$ or $V_*(x) = x^2$.

D. Cauchy oscillator

Our preceding discussion has actually been related to the generalization of the reverse engineering problem of Ref. [7]: (i) with an invariant pdf $\rho_*(x)$ in hands, derive $b(x)$ for the associated Langevin equation to reconstruct the Langevin-drien dynamics of the pdf, (ii) from the same invariant pdf $\rho_*(x)$, deduce the topologically - driven potential $\mathcal{V}(x)$ for the related Cauchy semigroup dynamics (thus granting that pdf an asymptotic pdf status).

In the present section we invert the reasoning and effectively follow the direct engineering route: having $\mathcal{V}(x)$,

we employ the semigroup dynamics principles to infer an asymptotic target $\rho_*(x)$. This construction will be supplemented by identifying, if any, the drift function $b(x)$ for the associated Langevin process that gives rise to the same asymptotic target.

To this end it seems natural to resort to simplest possible potential functions. Our choice (motivated by a simplicity of involved Fourier transforms and an immediate comparison with the Gaussian OU process of Section II.A) is the familiar harmonic oscillator potential $\mathcal{V}(x) = \frac{\kappa}{2} x^2 - \mathcal{V}_0$, constant \mathcal{V}_0 is left unspecified.

Our major object of interest is thus a pseudo-differential Hamiltonian (this is a very special, massless version, of the well known Hamiltonian for the relativistic harmonic oscillator problem, [12, 13]):

$$\hat{H}_{1/2} \equiv \lambda |\nabla| + \left(\frac{\kappa}{2} x^2 - \mathcal{V}_0 \right). \quad (23)$$

That in turn is a Cauchy analog of the familiar harmonic oscillator Hamiltonian of Section II.A:

$$\hat{H} = -D\Delta + \left(\frac{\gamma^2 x^2}{4D} - \frac{\gamma}{2} \right). \quad (24)$$

Since the quadratic function is admissible [10] as a semigroup potential, we need not to bother about an asymptotic approach towards an invariant density, but follow a direct reconstruction route: given \mathcal{V} of Eq. (24), deduce an invariant pdf ρ_* . To this end we turn back to Eq. (11) with $\mu = 1$ and consider a pseudo-differential equation

$$\left(\frac{\kappa}{2} x^2 - \mathcal{V}_0 \right) \rho_*^{1/2} = -\lambda |\nabla| \rho_*^{1/2} \quad (25)$$

to solve it with respect to ρ_* .

We denote $\tilde{f}(p)$ the Fourier transform of $f = \rho_*^{1/2}(x)$. Accordingly, Eq. (25) takes the following form:

$$-\frac{\kappa}{2} \Delta_p \tilde{f} + \gamma |p| \tilde{f} = \mathcal{V}_0 \tilde{f} \quad (26)$$

which, up to constants adjustments, can be recognized as the eigenvalue problem for the Schrödinger operator with the linear confining (modulus of the argument) potential, [22], see also [23, 24, 25]. Albeit with respect to momentum - space (here, wave-vector) variables, i.e. with $\Delta_p = d^2/dp^2$ replacing the conventional spatial Laplacian.

By changing an independent variable p to $k = (p - \sigma)/\zeta$, next denoting $\psi(k) = \tilde{f}(p)$ with the identifications $\sigma = \mathcal{V}_0/\gamma$ and $\zeta = (\kappa/2\gamma)^{1/3}$, we may rewrite the above eigenvalue problem (with \mathcal{V}_0 standing for an eigenvalue) in the form of the following ordinary differential equation

$$\frac{d^2 \psi(k)}{dk^2} = |k| \psi(k), \quad (27)$$

whose solutions can be represented in terms of Airy functions. A brief resume of how to deduce the eigenfunctions and eigenvalues of the original problem (26), is relegated

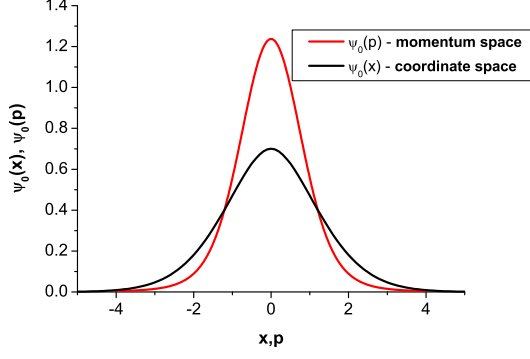


FIG. 6: Fourier image $\psi_0(k)$ of the function $\psi_0(x)$ along with this function

to Appendix. A unique normalized ground state function of the problem (26), (27) is composed of two Airy pieces that are glued together at the first zero y_0 of the Airy function derivative:

$$\psi_0(k) = A_0 \begin{cases} \text{Ai}(-y_0 + k), & k > 0 \\ \text{Ai}(-y_0 - k), & k < 0, \end{cases}$$

$$A_0 = [\text{Ai}(-y_0)\sqrt{2y_0}]^{-1}, \quad y_0 \approx 1.01879297. \quad (28)$$

The function (28) is a square root of the pdf in momentum space. We reproduce its shape in Fig. 6.

To transform the ground state solution back to coordinate space, we evaluate the inverse Fourier transformation of the ground state solution (28), see Appendix C for details. This yields the following real ground state wave function $f(x) \rightarrow \psi_0(x)$

$$\psi_0(x) = \frac{A_0}{\pi} \int_{-y_0}^{\infty} \text{Ai}(t) \cos x(t + y_0) dt = \rho_*^{1/2}(x), \quad (29)$$

which determines an invariant pdf $\rho_*(x)$ of the direct engineering problem (25) as follows:

$$\rho_*(x) = \left(\frac{A_0}{\pi}\right)^2 \left[\int_{-y_0}^{\infty} \text{Ai}(t) \cos x(t + y_0) dt \right]^2 \quad (30)$$

$$\equiv \left(\frac{A_0}{\pi}\right)^2 \int_{-y_0}^{\infty} dt \int_{-y_0}^{\infty} dt_1 \text{Ai}(t) \text{Ai}(t_1) \times$$

$$\times \cos x(t + y_0) \cos x(t_1 + y_0).$$

Even with an exact analytic formula for the normalized $\int_{-\infty}^{\infty} \rho_*(x) dx = 1$ function $\rho_*(x)$ (30) in hands, a better insight into its properties is achieved only by means of numerical methods. We depict both x and p -space versions of ψ_0 in Fig. 6.

In Fig. 7, the resultant invariant density $\rho_*(x)$ is represented by a full line. For comparison, we have depicted the Gaussian function with the same variance

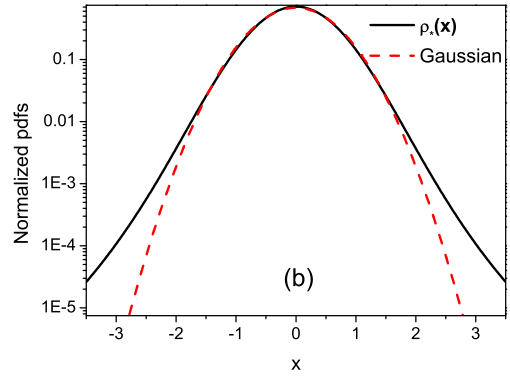
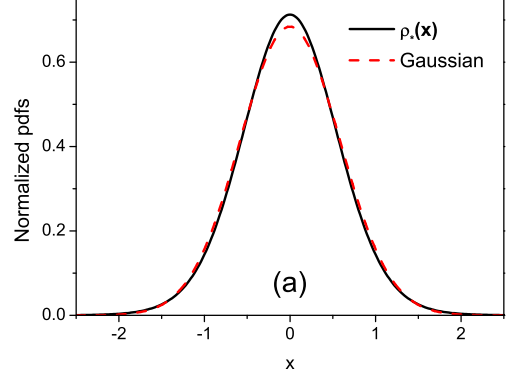


FIG. 7: Normalized invariant pdf (30) (full line) for the quadratic semigroup potential. The Gaussian function, centered at $x = 0$ and with the same variance $\sigma^2 = 0.339598$ is shown for comparison. Panel (a) shows functions in linear scale, while panel (b) shows them in logarithmic scale to better visualize their different behavior.

$\sigma^2 = 0.339598$, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$. It is shown by a dashed line. Fig. 7b shows both above functions in the log scale. It is clearly seen, that Airy-induced $\rho_*(x)$ decays at infinities slower than Gaussian.

Let us finally note, that the potential $\Phi(x)$ may be recovered from the ground state function (30) as the asymptotic pdf for topologically driven (semigroup) process has the form $\exp(-\Phi)$. Namely, $\Phi(x) \propto -\ln \rho_*(x)$, where $\rho_*(x)$ is given by Eq. (30). The inversion of the y axis of Fig. 7b shows this potential (compared with harmonic one $y = x^2$).

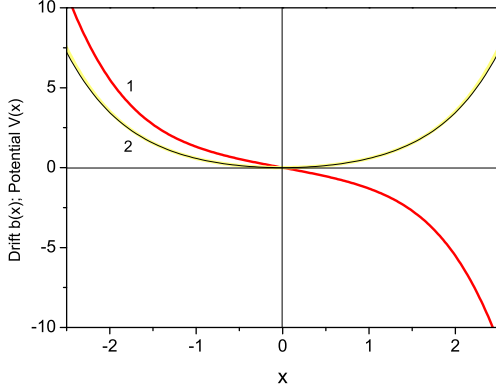


FIG. 8: Langevin - type drift $b(x)$ (curve 1) and its (force) potential $V(x)$ (curve 2), that give rise to an invariant density (30).

E. Reverse engineering for the Cauchy oscillator ground state pdf

For a given ρ_* the definition of a drift function $b(x)$ (we put either $\lambda = 1$ or define $b \rightarrow b/\lambda$) is:

$$b(x) = -\frac{1}{\rho_*(x)} \int [|\nabla| \rho_*(x)] dx \equiv \quad (31)$$

$$\frac{1}{\pi \rho_*(x)} \int dx \int_{-\infty}^{\infty} \frac{\rho_*(x+y) - \rho_*(x)}{y^2} dy.$$

Inserting $\rho_*(x)$, Eq. (30), we get

$$b(x) = -\frac{\int_{-y_0}^{\infty} \text{Ai}(t) \sin x(t+y_0) dt}{\int_{-y_0}^{\infty} \text{Ai}(t) \cos x(t+y_0) dt}. \quad (32)$$

The final formula for $b(x)$, (32), together with that for the corresponding Langevin force potential $V(x) \equiv -\int b(x) dx$ look a bit clumsy. Therefore it is appropriate to reiterate to numerics, see Fig. 8. The plot of $b(x)$ is reported in Fig.8 along with potential function $V(x) \equiv -\int b(x) dx$.

We were unable to determine the asymptotic of $b(x)$ at spatial infinities. It is possible, however, to expand Eq. (32) in power series at small x . It turns out, that these truncated series describe the function $b(x)$ (and correspondingly $V(x)$) surprisingly well.

To obtain this approximation, we expand both numerator and denominator of Eq. (32) in power series to obtain $\int_{-y_0}^{\infty} \text{Ai}(t) \sin x(t+y_0) dt = 0.824278x - 0.503237x^3 + 0.205648x^5 - 0.0650381x^7 + \dots$ for numerator and $\int_{-y_0}^{\infty} \text{Ai}(t) \cos x(t+y_0) dt = 0.809073 - 0.687715x^2 + 0.334243x^4 - 0.118792x^6 + 0.0339885x^8 - \dots$ for denominator. The series can be easily continued for larger amount

of terms. Now, the ratio of these series can be also expressed in the form of the series. Here, we reproduce only the truncated (up to x^5) version of the series

$$\begin{aligned} b(x) &\approx -1.01879x - 0.243986x^3 - 0.04056x^5, \\ V(x) &\approx 0.509395x^2 + 0.0609965x^4 + 0.00676x^6. \end{aligned} \quad (33)$$

The approximations (33) can be used to calculate numerically the Langevin-type dynamics with the invariant density (30).

IV. CONCLUSIONS

The Lévy - Schrödinger semigroup modeling sets a (mathematically rigorous) link between Lévy flights and pseudo - differential Hamiltonian systems. It is well known that the pdf of a free Lévy flight has so-called heavy tails and consequently does not have second and higher moments. Properly tailored external inhomogeneities are capable of "taming" a Lévy flight so that the resultant pdf admits higher moments which, in turn, makes possible to use such functions for the description of real physical (and other, like biological or economic) systems.

We have encoded the overall impact of inhomogeneities in the semigroup potential notion, that may be interpreted as an external potential in the affiliated pseudo-differential Hamiltonian operator. However, not any conceivable semigroup potential and thus, not any conceivable inhomogeneity, can make the corresponding jump-type process a mathematically well-behaved construction, where the Markovian dynamics entails an approach towards a unique stationary pdf.

In the present paper, we have investigated the behavior of Lévy oscillators in different external confining potentials. Our analysis shows that to control the long - time asymptotics of the above Lévy oscillator pdfs, suitable restrictions upon the Hamiltonian and its (semigroup or topologically-induced) potential $\mathcal{V}(x)$ need to be observed, [10].

Namely, $\mathcal{V}(x)$ should allow to be made positive (this is achieved by simple vertical shift of the entire function), should be locally bounded and needs to be measurable, i.e. should have a possibility to be approximated with arbitrary precision by step functions sequences. The fulfilment of these requirements provides verifiable criteria for an invariant pdf to be actually a time - asymptotic pdf of a semigroup (equivalently, topologically)-driven process.

A technical advantage of a semigroup formalism is the possibility of eigenfunction expansions for \hat{H} which allows to deduce explicit formulas for transition pdfs, [3]. As a byproduct of the above procedure we have completely solved the eigenvalue problem for the Hamiltonian operator of so-called Cauchy oscillator in terms of Airy functions. The ground state wave function of such oscillator has been obtained analytically. Its square defines the invariant pdf of Cauchy oscillator process. The latter pdf is approached by the jump-type process at large times.

We have extended the targeted stochasticity problem of Ref. [7] to the above semigroup-driven (topological) Lévy processes, which are widely used in literature to model various systems, like polymers, glasses and complex networks. Our departure point was as follows: having an invariant pdf $\rho_*(x)$, recover not only the Langevin drift $b(x)$ and potential $V(x) = -\int b(x)dx$, but also the potential $\mathcal{V}(x)$ of the corresponding topological (semi-group) Lévy process, being attributed to the same invariant pdf.

Furthermore, we have relaxed a common pdf requirement and have reformulated the targeted stochasticity problem as a task of reproducing a suitable contractive semigroup, given an invariant pdf, with the Lévy (specifically, Cauchy) driver in action. We have shown, that the semigroup modeling provides much stronger confining properties than the standard Langevin one, such that the resultant asymptotic pdf may have all moments.

To be more specific, if both above approaches involve (albeit differently) the same conservative force potential $\Phi = V_*(x)$, then the asymptotic pdf in the Langevin scenario, being an inverse polynomial, has no more than a finite number (the degree of the polynomial minus 2) of moments. At the same time, the corresponding asymptotic pdf of the topological process, being of the Gibbs form $\propto \exp(-\Phi)$, may in principle admit all moments. If both mechanisms refer to a common asymptotic pdf, the latter being derivable in the Langevin approach, the previous statement is no longer valid, [4].

It turns out that the asymptotic behavior of a time-dependent pdf, in the semigroup (topological) modeling, may critically depend on the initial data choice, like e.g. the location of the initial pdf in the "potential landscape" of $V_*(x)$. The signature of such behavior is encoded in the functional form of a semigroup potential $\mathcal{V}(x)$, derived from the a priori chosen invariant pdf $\rho_*(x)$, by means of above generalized reverse engineering procedure. If the effective (semigroup) potential obeys the requirements of [10], we may expect that a prescribed invariant density is indeed approached in the large time asymptotic of the random process, irrespective of the initial (pdf) data choice. These requirements need to be verified for each specific guess about a functional form of the prescribed invariant pdf ρ_* .

APPENDIX A: SCHRÖDINGER EIGENVALUE PROBLEM FOR THE LINEAR POTENTIAL AND AIRY FUNCTION

In the present Appendix we briefly recapitulate, basically retrievable in the literature but not accessible in minute detail nor in a closed form, (see, e.g. [22]), a procedure of construction of the eigenfunctions of equation (27). Symmetry arguments (see, e.g. [22]) and an explicit solution of corresponding Schrödinger equation [25] lead to the following expression for the eigenfunctions (here

we substitute y for k)

$$\psi_n(y) = \begin{cases} A_n \text{Ai}(-y_n + y), & y > 0 \\ \pm A_n \text{Ai}(-y_n - y), & y < 0, \end{cases} \quad (\text{A1})$$

where n enumerates eigenvalues (and corresponding eigenfunctions). In Eq. (A1), y_n numbers the n -th zero of the function $\text{Ai}(y)$ (for odd n), or of its derivative for even n (observe that zeros of both function Ai and its derivative lie on the negative semi-axis). A_n is a normalization coefficient, determined by the standard identity

$$A_n^2 \int_{-\infty}^{\infty} \psi_n^2(y) dy = 1. \quad (\text{A2})$$

Here we use simply ψ^2 (rather than $|\psi|^2 \equiv \psi\psi^*$) since the eigenfunctions are real.

The method of construction of the above wave functions from initial Airy function is shown graphically in Fig.9. We link either function (odd n - Fig.9b) or its derivative (even n - Fig.9a) in one of the zeros. In other words, the wave function of n -th state is created by shifting the Airy function on the positive half-axis to the right so that its n -th zero y_n coincides with the origin (zero) of the coordinate system. After this step we continue a function to the negative half-axis either evenly (for even n) or oddly for odd n . The expression for A_n is as follows

$$A_n^2 \left[\int_{-\infty}^0 \text{Ai}^2(-y_n - y) dy + \int_0^{\infty} \text{Ai}^2(-y_n + y) dy \right] = 1. \quad (\text{A3})$$

The evaluation of integrals (A3) yields after some algebra

$$A_n = \frac{1}{\sqrt{2I_2}} \equiv \frac{1}{\sqrt{2}} \left[y_n \text{Ai}^2(-y_n) + \text{Ai}'^2(-y_n) \right]^{-1/2}, \quad (\text{A4})$$

where prime means derivative with respect to an argument. Further simplifications of Eq. (A4) are possible if we observe that for even n $\text{Ai}'^2(-y_n) = 0$ and for odd n $\text{Ai}^2(-y_n) = 0$ so that

$$A_n = \begin{cases} [\text{Ai}'(-y_n)\sqrt{2}]^{-1}, & n \text{ is odd} \\ [\text{Ai}(-y_n)\sqrt{2y_n}]^{-1}, & n \text{ is even.} \end{cases} \quad (\text{A5})$$

Also, the corresponding (dimensional) energy eigenvalues $E_n \equiv \mathcal{V}_{0n}$ can be found from a condition that in the p space all p_n corresponding to zeros y_n must be zero. In other words, $x_n = y_n\zeta + \sigma \equiv 0$ or

$$E_n = -\gamma\zeta y_n \equiv |y_n| \left(\frac{\kappa}{2\gamma} \right)^{1/3}. \quad (\text{A6})$$

Here we reflect the fact that zeros of the Airy function and its derivative are negative.

Now we are in a position to write the explicit form of several first wave functions of Eq. (27). The ground state function is defined by the Eq. (28). The first excited

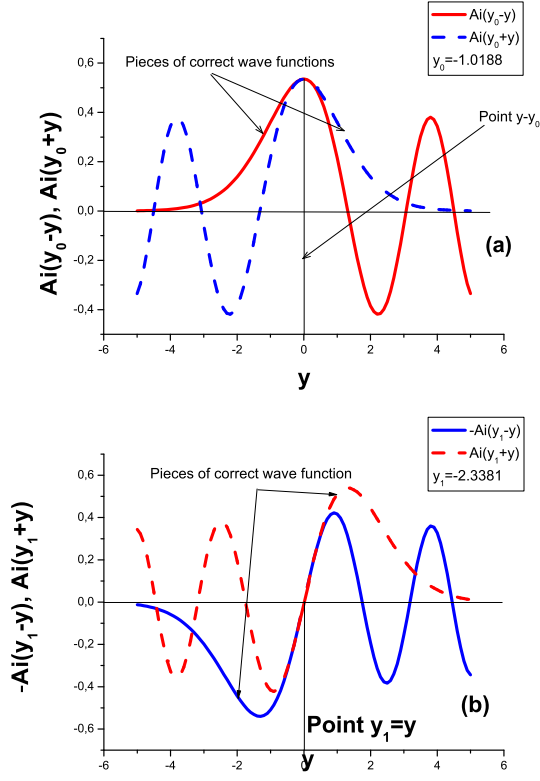


FIG. 9: The idea of the wave functions construction; a - ground state (and even n), b - first excited state (and odd n)

state ($n = 1$) has the form

$$\psi_1(y) = A_1 \begin{cases} \text{Ai}(-y_1 + y), & y > 0 \\ -\text{Ai}(-y_1 - y), & y < 0, \end{cases} \quad (\text{A7})$$

$$A_1 = [\text{Ai}'(-y_1)\sqrt{2}]^{-1}, \quad y_1 \approx 2.3381.$$

For the second excited state ($n = 2$) we get

$$\psi_2(y) = A_2 \begin{cases} \text{Ai}(-y_2 + y), & y > 0 \\ \text{Ai}(-y_2 - y), & y < 0, \end{cases} \quad (\text{A8})$$

$$A_2 = [\text{Ai}(-y_2)\sqrt{2y_2}]^{-1}, \quad y_2 \approx 3.2482.$$

The third excited state ($n = 3$) reads

$$\psi_3(y) = A_3 \begin{cases} \text{Ai}(-y_3 + y), & y > 0 \\ -\text{Ai}(-y_3 - y), & y < 0, \end{cases} \quad (\text{A9})$$

$$A_1 = [\text{Ai}'(-y_3)\sqrt{2}]^{-1}, \quad y_3 \approx 3.2482.$$

The functions (28), (A7) - (A9) are plotted in Fig. 10. The conformance with oscillation theorem (the n -th wave function of a discrete spectrum can have only n zeros on its domain) [25] is seen.

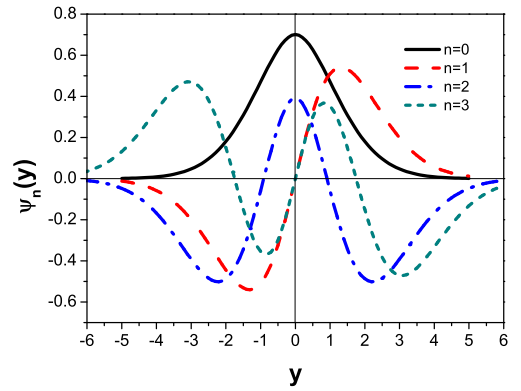


FIG. 10: Several wave functions for the linear modulus potential, Eq. (27).

APPENDIX B: THE APPROXIMATE EXPRESSIONS FOR ZEROES OF AIRY FUNCTION AND ITS DERIVATIVE

For large negative y the function $\text{Ai}(y)$ has the following asymptotic expansion

$$\text{Ai}(x \rightarrow -\infty) \approx -\cos \left[\frac{\pi}{4} + \frac{2x\sqrt{-x}}{3} \right] \frac{(-1)^{3/4} \left(\frac{1}{x} \right)^{1/4}}{\sqrt{\pi}} - \frac{5(-1)^{1/4} \left(\frac{1}{x} \right)^{7/4}}{48\sqrt{\pi}} \sin \left[\frac{\pi}{4} + \frac{2x\sqrt{-x}}{3} \right]. \quad (\text{B1})$$

Equating series (B1) to zero, we obtain the desired analytical expression for zeros of the Ai function. We observe that the coefficient before \sin , proportional to $x^{-7/4}$, de-

cays at infinity much faster than that preceding the cosine. So, we simply equate to zero the argument of cosine

function, getting

$$y_n = -\left(\frac{3\pi}{2}\right)^{2/3} \left(n + \frac{3}{4}\right)^{2/3}. \quad (\text{B2})$$

It turns out that (B2) gives a fairly good approximation for zeros that begin at the lowest eigenvalue $n = 1$. We compare exact and approximate (B2) solutions:

$$\begin{aligned} n=0 \quad y_0^{\text{exact}} &= -2.3381, \quad y_0^{\text{appr}} = -2.32025, \\ n=1 \quad y_1^{\text{exact}} &= -4.0879, \quad y_1^{\text{appr}} = -4.08181, \\ n=2 \quad y_2^{\text{exact}} &= -5.5206, \quad y_2^{\text{appr}} = -5.51716, \\ &\dots\dots\dots \\ n=8 \quad y_8^{\text{exact}} &= -11.0085, \quad y_8^{\text{appr}} = -11.0077, \\ n=9 \quad y_9^{\text{exact}} &= -11.936, \quad y_9^{\text{appr}} = -11.9353. \end{aligned} \quad (\text{B3})$$

The same asymptotic analysis can be performed for the derivative of Airy function. We end up with

$$y_n^{(+)} = -\left(\frac{3\pi}{2}\right)^{2/3} \left(n + \frac{1}{4}\right)^{2/3}. \quad (\text{B4})$$

A comparison of exact and approximate roots goes as follows

$$\begin{aligned} n=0 \quad y_0^{(+)\text{exact}} &= -1.0188, \quad y_0^{(+)\text{appr}} = -1.11546, \\ n=1 \quad y_1^{(+)\text{exact}} &= -3.2482, \quad y_1^{(+)\text{appr}} = -3.26163, \\ n=2 \quad y_2^{(+)\text{exact}} &= -4.8201, \quad y_2^{(+)\text{appr}} = -4.82632, \\ &\dots\dots\dots \\ n=8 \quad y_8^{(+)\text{exact}} &= -11.4751, \quad y_8^{(+)\text{appr}} = -11.4762, \\ n=9 \quad y_9^{(+)\text{exact}} &= -12.3848, \quad y_9^{(+)\text{appr}} = -12.3857. \end{aligned} \quad (\text{B5})$$

APPENDIX C: FOURIER IMAGES OF THE WAVE FUNCTIONS

Since our departure point in Section III D was the Schrödinger-type eigenvalue problem in momentum space, it turns out to be useful to discuss Fourier images of the above eigenfunctions. We recall that the inverse

(from momentum to coordinate space) Fourier transform is defined as $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(p) e^{-ipx} dp$.

For the ground state we proceed accordingly. We substitute Eq. (28) into the Fourier integral to obtain

$$\psi_0(x) = \frac{A_0}{2\pi} (I_1 + I_2), \quad (\text{C1})$$

$$I_1 = \int_{-\infty}^0 \text{Ai}(-y_0 - p) e^{ipx} dp = \int_{-y_0}^{\infty} \text{Ai}(t) e^{-ix(t+y_0)} dt,$$

$$I_2 = \int_0^{\infty} \text{Ai}(-y_0 + p) e^{ipx} dp = \int_{-y_0}^{\infty} \text{Ai}(t) e^{-ix(t+y_0)} dt,$$

$$I_1 + I_2 = \int_{-y_0}^{\infty} \text{Ai}(t) \left[e^{-ix(t+y_0)} + e^{ix(t+y_0)} \right] dt.$$

Hence, the Fourier image of the ground state wave function is determined by the above equation (29). For the higher even n the above method yields

$$\psi_{\text{even}}(x) = \frac{A_n}{\pi} \int_{-y_n}^{\infty} \text{Ai}(t) \cos x(t + y_n) dt, \quad (\text{C2})$$

For odd states

$$\psi_{\text{odd}}(p) = \frac{A_n}{2\pi} (I_2 - I_1), \quad (\text{C3})$$

$$I_1 \equiv \int_{-\infty}^0 \text{Ai}(-y_n - p) e^{ipx} dp = \int_{-y_n}^{\infty} \text{Ai}(t) e^{-ip(t+y_n)} dt,$$

$$I_2 \equiv \int_0^{\infty} \text{Ai}(-y_n + p) e^{ipx} dp = \int_{-y_n}^{\infty} \text{Ai}(t) e^{ip(t+y_n)} dt,$$

$$I_2 - I_1 = A_n \int_{-y_n}^{\infty} \text{Ai}(t) \left[e^{ip(t+y_n)} - e^{-ip(t+y_n)} \right] dt.$$

In other words, the Fourier images of odd wave functions have the form

$$\psi_{\text{odd}}(p) = i \frac{A_n}{\pi} \int_{-y_n}^{\infty} \text{Ai}(t) \sin p(t + y_n) dt. \quad (\text{C4})$$

Here, A_n are determined by Eqs. (A5). It is seen that odd Fourier images are imaginary odd functions. This (imaginary coefficient) does not affect the physical meaning of $|\psi_{\text{odd}}(p)|^2$ which is a probability density.

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